

UNITARY EQUIVALENCE TO TRUNCATED TOEPLITZ OPERATORS

E. STROUSE, D. TIMOTIN, AND M. ZARRABI

ABSTRACT. In this paper we investigate operators unitarily equivalent to truncated Toeplitz operators. We show that this class contains certain sums of tensor products of truncated Toeplitz operators. In particular, it contains arbitrary inflations of truncated Toeplitz operators; this answers a question posed in [4].

1. INTRODUCTION

Truncated Toeplitz operators, defined on certain subspaces of the H^2 called *model spaces*, are a generalization of the operators associated with Toeplitz matrices. They are introduced and discussed in great detail in a recent survey paper by Sarason [10]. Some special cases have appeared long ago in the literature: the model operators for contractions with defect number one as well as their commutant are truncated Toeplitz operators with analytic symbols (see, for instance, [9, 11, 8]). This is a new area of study, and many simple questions remain open. The basic reference for this subject is [10], subsequent work is done in [4, 6, 3, 2].

The truncated Toeplitz operators live on the model spaces $K_\Theta = H^2 \ominus \Theta H^2$, with Θ an inner function. These are the subspaces of H^2 invariant for the adjoint of multiplication by the variable z (see Section 2 for precise definitions). Given a model space K_Θ and a function $\phi \in L^2$, the truncated Toeplitz operator A_ϕ^Θ is defined on a dense subspace of K_Θ as the compression to K_Θ of multiplication by ϕ . The function ϕ is then called a symbol of the operator. For $\Theta = z^n$ one obtains the usual Toeplitz matrices, but consideration of such operators for arbitrary inner functions brings up many new and interesting questions.

In this paper we study unitary equivalence to truncated Toeplitz operators. This problem is discussed at length in [4], where it is shown, in particular, that all rank one operators, two by two matrices, and normal operators are unitarily equivalent to truncated Toeplitz operators. Thus the class of operators unitarily equivalent to truncated Toeplitz operators is larger than might be expected. In [4] the authors ask several questions concerning this class; in particular, whether it contains inflations and direct sums of truncated Toeplitz operators.

2000 *Mathematics Subject Classification.* 47B35, 47B32, 47A45.

Key words and phrases. Model spaces, truncated Toeplitz operators, unitary equivalence.

The purpose of this paper is to continue along this line of investigation. By obtaining new classes of operators unitarily equivalent to truncated Toeplitz operators, we answer in particular a question in [4] concerning inflations, and provide some partial results related to direct sums. The methods used are mostly based on analyzing the action of composition by inner functions on model spaces and on truncated Toeplitz operators. It is suggested by the proof of Theorem 5.8 in [4]; a similar technique appears, in the case of Toeplitz operators on H^2 , in a much earlier paper of Cowen [7]. It should be noted that little is known about unitary equivalence to classical Toeplitz operators or to Toeplitz matrices.

The plan of the paper is the following: the next section is devoted to preliminaries. We introduce the notations and obtain some basic results concerning the effect on model spaces of composition by an inner function. Section 3 contains the main result, which is applied to some particular situations in Section 4. Finally, Section 5 is devoted to a special type of direct sum, investigating when the direct sum of a truncated Toeplitz operator with the zero operator is unitarily equivalent to a truncated Toeplitz operator.

2. MODEL SPACES AND COMPOSITION

We start by introducing the main objects and results that we shall use; convenient references are [8] for model spaces and [10] for TTOs.

Let H^2 be the classical Hardy space of holomorphic functions on the open unit disc \mathbb{D} having square-summable Taylor coefficients at the origin. By considering radial limits almost everywhere with respect to Lebesgue measure dm on $\partial\mathbb{D} = \mathbb{T}$, H^2 can be identified with the closed subspace of $L^2 = L^2(\mathbb{T}, dm)$ consisting of functions with negative Fourier coefficients equal to 0; its orthogonal complement $L^2 \ominus H^2$ is denoted by H_-^2 . An element $\Theta \in H^2$ is called an inner function if $|\Theta(e^{it})| = 1$ for almost all $t \in [0, 2\pi]$; an inner function Θ is of order n if Θ is a finite Blaschke product of degree n , and of order infinity otherwise.

Let S denote multiplication by z on H^2 . The closed subspaces of H^2 invariant for the adjoint S^* of S are precisely of the form $K_\Theta = H^2 \ominus \Theta H^2$, with Θ inner; they are often called *model spaces*. In general we will use the notation P_Y for the orthogonal projection onto a given subspace Y of L^2 ; we abbreviate P_{H^2} to P_+ , $P_{(H^2)^\perp}$ to P_- , and P_{K_Θ} to P_Θ . The map $f \mapsto J(f) = \Theta \bar{z} \bar{f}$ is an anti-isometry of K_Θ onto itself; in particular, we will use in the proof of Proposition 2.3 the fact that if $f \in K_\Theta$, then $\Theta \bar{f} \in zH^2$.

For $\phi \in L^2$, the (bounded or unbounded) *truncated Toeplitz operator* (TTO) A_ϕ^Θ is defined on K_Θ by $A_\phi^\Theta f = P_\Theta \phi f$. Note that since $H^\infty \cap K_\Theta$ is dense in K_Θ ([1], see also [5]), A_ϕ^Θ always has a dense domain. The function ϕ is called a *symbol* of the operator. It is not uniquely defined; more precisely,

$$A_\phi^\Theta = A_\psi^\Theta \iff \phi - \psi \in \Theta H^2 + \overline{\Theta H^2}. \quad (2.1)$$

Since tensor products appear quite often in the paper, we will use a different notation for rank one operators, namely

$$(x \odot y)(\xi) = \langle \xi, y \rangle x.$$

For $\lambda \in \mathbb{D}$ one defines

$$k_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z}; \quad \tilde{k}_\lambda^\Theta(z) = (J(k_\lambda^\Theta))(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}. \quad (2.2)$$

The functions k_λ^Θ are the reproducing kernels in K_Θ for $\lambda \in \mathbb{D}$. In the case when the angular derivative of Θ at $\zeta \in \mathbb{T}$ exists, then formulas (2.2), with λ replaced by ζ , define functions in K_Θ , evaluation in ζ is a continuous functional on K_Θ , and k_ζ^Θ is the corresponding reproducing kernel.

The reproducing kernels produce the rank one TTOs in K_Θ . More precisely, it is shown in [10, Theorem 5.1] that these are exactly the scalar multiples of $\tilde{k}_\lambda^\Theta \odot k_\lambda^\Theta$, $k_\lambda^\Theta \odot \tilde{k}_\lambda^\Theta$ for $\lambda \in \mathbb{D}$, and $k_\zeta^\Theta \odot k_\zeta^\Theta$ for the points $\zeta \in \mathbb{T}$ where the angular derivative of Θ exists. We will use below the precise form of some symbols (which appears also in [10]): namely, $\phi = \bar{z}\Theta$ is a symbol for $\tilde{k}_0^\Theta \odot k_0^\Theta$ and, for $\zeta \in \mathbb{T}$, $\phi = k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1$ is a symbol for $k_\zeta^\Theta \odot k_\zeta^\Theta$.

Lemma 2.1. *Suppose that Θ is an inner function and that $\phi \in L^2$. If A_ϕ^Θ is bounded, then $A_{z^j\phi}^\Theta$ is also bounded for any $j \in \mathbb{Z}$.*

Proof. Suppose first that $j = 1$. Denote by P_Θ^0 the projection onto $K_\Theta \ominus \mathbb{C}\tilde{k}_0^\Theta$. Take $f \in K_\Theta$. We have

$$A_{z\phi}^\Theta f = A_{z\phi}^\Theta (P_\Theta^0 f + (I - P_\Theta^0)f) = P_\Theta(z\phi P_\Theta^0 f) + P_\Theta(z\phi(I - P_\Theta^0)f). \quad (2.3)$$

One checks easily that, if $g \in K_\Theta$ and $g \perp \tilde{k}_0^\Theta$, then $zg \in K_\Theta$. Therefore $zP_\Theta^0 f \in K_\Theta$, and thus

$$P_\Theta(z\phi P_\Theta^0 f) = P_\Theta\phi(zP_\Theta^0 f) = A_\phi^\Theta(zP_\Theta^0 f).$$

Since A_ϕ^Θ is bounded by assumption, $P_\Theta z\phi P_\Theta^0$ is bounded. On the other hand, $(I - P_\Theta^0)$, the projection of rank 1 onto $\mathbb{C}\tilde{k}_0^\Theta$, has the formula $h_0^\Theta \odot h_0^\Theta$, where we have denoted $h_0^\Theta = \frac{\tilde{k}_0^\Theta}{\|\tilde{k}_0^\Theta\|}$. Since $z\phi h_0^\Theta \in L^2$, $z\phi h_0^\Theta \odot h_0^\Theta$ is a bounded rank one operator, and we may write

$$P_\Theta z\phi(I - P_\Theta^0)f = P_\Theta z\phi\langle f, h_0^\Theta \rangle h_0^\Theta = P_\Theta(z\phi h_0^\Theta \odot h_0^\Theta)f.$$

It follows then from (2.3) that $A_{z\phi}^\Theta$ is bounded.

The statement is thus proved for $j = 1$. It is obvious that we may apply induction to obtain $A_{z^j\phi}^\Theta$ bounded for any $j \in \mathbb{N}$. But if A_ϕ^Θ is bounded then $A_\phi^\Theta = (A_\phi^\Theta)^*$ is bounded, and therefore also $A_{z^j\phi}^\Theta = (A_{z^{-j}\phi}^\Theta)^*$ for $j \leq 0$. The lemma is proved. \square

The result of the next proposition is implicitly contained in the proof of [7, Theorem 1].

Proposition 2.2. *Suppose B is an inner function. Then the formula*

$$h \otimes f \mapsto h(f \circ B)$$

can be extended linearly to define a unitary operator Ω_B from $K_B \otimes L^2$ onto L^2 . The operator Ω_B maps $K_B \otimes H^2$ onto H^2 .

Proof. To show that the given map is isometric, it is enough to check that it maps an orthonormal basis of $K_B \otimes L^2$ to an orthonormal family of L^2 . Let $(h_i)_{i \in I}$ be an orthonormal basis for K_B ; then $(h_i \otimes z^j)_{i \in I, j \in \mathbb{Z}}$ is a basis for $K_B \otimes L^2$, and we have to show that

$$\langle h_i B^j, h_{i'} B^{j'} \rangle = \delta_{i,i'} \delta_{j,j'}.$$

If $j = j'$, then

$$\langle h_i B^j, h_{i'} B^j \rangle = \delta_{i,i'}$$

since multiplication by an inner function is isometric. On the other hand, if $j \neq j'$, assume, for instance, that $j < j'$; then

$$\langle h_k B^j, h_{k'} B^{j'} \rangle = \langle h_k, h_{k'} B^{j'-j} \rangle$$

which equals zero since the first term is in K_B , while the second is in BH^2 .

We have $\Omega_B(K_B \otimes \mathbb{C}z^j) = B^j K_B$. Since $L^2 = \bigoplus_{j \in \mathbb{Z}} B^j K_B$, it follows that Ω_B is onto and thus unitary. Also, $H^2 = \bigoplus_{j \in \mathbb{N}} B^j K_B$, and thus Ω_B maps $K_B \otimes H^2$ precisely onto H^2 . \square

We will now consider a second inner function Θ . Remarkably, the unitary operator of Proposition 2.2 behaves well with respect to K_Θ .

Proposition 2.3. *Suppose that B and Θ are inner functions, while Ω_B is the unitary defined in Proposition 2.2. Then*

$$\Omega_B(K_B \otimes \Theta H^2) = (\Theta \circ B)H^2, \quad \Omega_B(K_B \otimes K_\Theta) = K_{\Theta \circ B}.$$

Proof. It is enough to prove the inclusions $\Omega_B(K_B \otimes \Theta H^2) \subset (\Theta \circ B)H^2$ and $\Omega_B(K_B \otimes K_\Theta) \subset K_{\Theta \circ B}$, since Ω_B is unitary and $H^2 = (\Theta \circ B)H^2 \oplus K_{\Theta \circ B}$.

If we fix $h \in K_B$ and $j \in \mathbb{N}$, we have

$$\Omega_B(h \otimes \Theta z^j) = h(\Theta \circ B)B^j \in (\Theta \circ B)H^2,$$

and thus the first inclusion is true.

For the second inclusion, let f be another element of K_Θ . We need to see that

$$\langle h(f \circ B), (\Theta \circ B)z^k \rangle = 0$$

for all $k \geq 0$. This equality can be written as

$$\langle S^{*k}h, (\Theta \bar{f}) \circ B \rangle = 0,$$

where S is the shift operator on H^2 ($Sf = z f$). But, if $f \in K_\Theta$, then $\Theta \bar{f} \in zH^2$, and therefore $(\Theta \bar{f}) \circ B \in BH^2$. Since $h \in K_B$ and $S^*K_B \subset K_B$, the scalar product above is indeed null. \square

We will denote by $\omega_B : K_B \otimes K_\Theta \rightarrow K_{\Theta \circ B}$ the restriction of Ω_B to $K_B \otimes K_\Theta$.

3. TRUNCATED TOEPLITZ OPERATORS

We start with a lemma about the action of certain multiplications.

Lemma 3.1. *Suppose that B is an inner function, $\psi, \phi \in L^2$, $h \in K_B \cap H^\infty$, $f \in L^\infty$, and that the operators $A_{\bar{B}^j \psi}^B$ are nonzero only for a finite number of $j \in \mathbb{Z}$. Then*

$$\psi(\phi \circ B)\Omega_B(h \otimes f) = \Omega_B\left(\sum_j (A_{\bar{B}^j \psi}^B h \otimes z^j \phi f)\right).$$

Proof. We have $\psi(\phi \circ B)\Omega_B(h \otimes f) = \psi(\phi \circ B)h(f \circ B)$. The assumptions imply that $\psi h \in L^2 = \bigoplus_{j \in \mathbb{Z}} B^j K_B$. Since

$$P_{B^j K_B}(\psi h) = B^j P_{K_B} \bar{B}^j \psi h = B^j A_{\bar{B}^j \psi}^B h,$$

and $A_{\bar{B}^j \psi}^B \neq 0$ only for a finite number of j , it follows that in the orthogonal decomposition $\psi h = \sum_j P_{B^j K_B}(\psi h)$ the sum is finite. Thus, we can write

$$\begin{aligned} \psi h(\phi \circ B)(f \circ B) &= \left(\sum_j P_{B^j K_B}(\psi h) \right) (\phi \circ B)(f \circ B) \\ &= \sum_j B^j P_B(\bar{B}^j \psi h)(\phi \circ B)(f \circ B) \\ &= \sum_j \Omega_B \left(A_{\bar{B}^j \psi}^B h \otimes z^j \phi f \right), \end{aligned}$$

which finishes the proof. \square

Theorem 3.2. *Let B and Θ be inner functions, and suppose that $\psi, \phi \in L^2$ are subjected to the following conditions:*

(a) *The operators $A_{\bar{B}^j \psi}^B$ are bounded and nonzero only for a finite number of $j \in \mathbb{Z}$.*

(b) *A_ϕ^Θ is bounded.*

(c) *$\psi(\phi \circ B) \in L^2$.*

Then $A_{\psi(\phi \circ B)}^{\Theta \circ B}$ is bounded, and

$$A_{\psi(\phi \circ B)}^{\Theta \circ B} \omega_B = \omega_B \left(\sum_j (A_{\bar{B}^j \psi}^B \otimes A_{z^j \phi}^\Theta) \right). \quad (3.1)$$

Proof. Let us note first that, since $\Omega_B(K_B \otimes K_\Theta) = K_{\Theta \circ B}$, we have

$$P_{\Theta \circ B} = \Omega_B(P_B \otimes P_\Theta) \Omega_B^*.$$

Let $h \in K_B \cap H^\infty$, $f \in K_\Theta \cap H^\infty$. We have, using Lemma 3.1,

$$\begin{aligned} A_{\psi(\phi \circ B)}^{\Theta \circ B} \omega_B(h \otimes f) &= P_{\Theta \circ B} \psi(\phi \circ B) \Omega_B(h \otimes f) = \Omega_B(P_B \otimes P_\Theta) \Omega_B^* \psi(\phi \circ B) \Omega_B(h \otimes f) \\ &= \Omega_B(P_B \otimes P_\Theta) \left(\sum_j (A_{B^j \psi}^B h \otimes z^j \phi f) \right) \\ &= \omega_B \left(\sum_j (A_{B^j \psi}^B h \otimes A_{z^j \phi}^\Theta f) \right). \end{aligned}$$

By Lemma 2.1, $A_{z^j \phi}^\Theta$ is bounded for any j . Therefore the operator on the right hand side of the last equation is bounded by assumption; it follows that $A_{\psi(\phi \circ B)}^{\Theta \circ B}$ is also bounded, and (3.1) is true. The theorem is proved. \square

Remark 3.3. (1) If $\psi \in B^k K_B$ for some $k \in \mathbb{Z}$, then $A_{B^j \psi}^B = 0$ for $j \neq k, k+1$, while $\psi(\phi \circ B) \in L^2$ by Proposition 2.2. Therefore, if $\psi \in \oplus_{j \in J} B^j K_B$ for a finite set J , then $A_{B^j \psi}^B$ is nonzero only for a finite number of $j \in \mathbb{Z}$, while $\psi(\phi \circ B) \in L^2$. Thus part of the hypotheses of Theorem 3.2 are automatically satisfied.

(2) The boundedness conditions in the statement of Theorem 3.2 are immediately satisfied if $\phi, \psi \in H^\infty$. It is known however [2, 3] that there exist bounded TTOs that have no bounded symbols, and therefore such an assumption would reduce the generality of the result. In [2] one characterizes precisely the inner functions Θ that have the property that any bounded TTO on K_Θ has a bounded symbol.

We will discuss in the next section several applications of Theorem 3.2; but first we give some simple but important consequences.

Corollary 3.4. *Suppose that Θ is an inner function, that $\phi \in L^2$, and that B is an inner function of order k for some $k = 1, 2, \dots, \infty$. Assume also that A_ϕ^Θ is bounded. Then $A_{\phi \circ B}^{\Theta \circ B}$ is bounded and unitarily equivalent to $I_k \otimes A_\phi^\Theta$.*

Proof. Take $\psi \equiv 1$. We have $A_\psi^B = I_{K_B}$ and $A_{B^j \psi}^B = 0$ for $j \neq 0$, whence (a) in Theorem 3.2 is satisfied. (b) is satisfied by assumption, while (c) is true since $\phi \circ B \in L^2$ by Littlewood's Subordination Theorem. Therefore $A_{\phi \circ B}^{\Theta \circ B}$ is bounded, and (3.1) becomes

$$A_{\phi \circ B}^{\Theta \circ B} \omega_B = \omega_B(I_{K_B} \otimes A_\phi^\Theta),$$

which proves the corollary. \square

For any Θ and k , if we take B to be an inner function of order k , then Corollary 3.4 answers Question 5.10 of [4].

The next immediate consequence is the analogue for TTOs of a result for usual Toeplitz operators from the paper [7] mentioned in the introduction.

Corollary 3.5. *Suppose that Θ is an inner function and that $\phi \in L^2$, such that A_ϕ^Θ is bounded. If B_1, B_2 are two inner functions of the same order, then $A_{\phi \circ B_1}^{\Theta \circ B_1}$ and $A_{\phi \circ B_2}^{\Theta \circ B_2}$ are bounded and unitarily equivalent.*

4. APPLICATIONS

We intend to apply Theorem 3.2 to different choices of B and/or Θ , in order to find several classes of operators which are unitarily equivalent to TTOs. In all the examples in this section we choose $\psi \in K_B$. Then the assumptions of Theorem 3.2 are satisfied (see Remark 3.3 (1)); moreover, $A_{B^j \psi}^B = 0$ for all $j \neq 0, 1$. Thus we have the following corollary of Theorem 3.2, which deserves to be mentioned separately.

Corollary 4.1. *If $\psi \in K_B$ and $A_\psi^B, A_{B\psi}^B, A_\phi^\Theta$ are bounded, then $A_{\psi(\phi \circ B)}^{\Theta \circ B}$ is bounded and*

$$A_{\psi(\phi \circ B)}^{\Theta \circ B} \omega_B = \omega_B [A_\psi^B \otimes A_\phi^\Theta + A_{B\psi}^B \otimes A_{z\phi}^\Theta]. \quad (4.1)$$

4.1. Suppose $B(z) = z^n$. Then K_B is equal to the set of analytic polynomials of degree at most $k-1$, and the monomials $\psi_m(z) = z^m$, $m = 0, \dots, k-1$ form a basis of K_B . If $A = (a_{ij})$ is the matrix of $A_{\psi_m}^B$ with respect to this basis, then

$$a_{ij} = \begin{cases} 1 & \text{if } i = j + m, \\ 0 & \text{otherwise,} \end{cases}$$

and $A_{B\psi_m}^B = (A_{\psi_{n-m}}^B)^*$.

Let $\phi_m \in L^2$, $m = 0, \dots, n-1$, such that all the operators $A_{\phi_m}^\Theta$ are bounded. Define

$$h(z) = \sum_{m=0}^{n-1} z^m (\phi_m(z^n)) = \sum_{m=0}^{n-1} z^m (\phi_m \circ B).$$

Using the natural identification of $K_B \otimes K_\Theta$ with the direct sum of n copies of K_Θ , formula (4.1) and the above remarks imply that $A_h^{\Theta \circ B} = \sum_{m=0}^{n-1} A_{z^m(\phi_m \circ B)}^{\Theta \circ B}$ is bounded and unitarily equivalent to the block Toeplitz operator matrix

$$\begin{pmatrix} A_{\phi_0}^\Theta & A_{z\phi_{n-1}}^\Theta & \dots & A_{z\phi_1}^\Theta \\ A_{\phi_1}^\Theta & A_{\phi_0}^\Theta & \dots & A_{z\phi_2}^\Theta \\ \ddots & \ddots & \ddots & \ddots \\ A_{\phi_{n-1}}^\Theta & A_{\phi_{n-2}}^\Theta & \dots & A_{\phi_0}^\Theta \end{pmatrix}$$

It is easily seen that, for $k < \infty$, Corollary 3.4 is a particular case of the above formula.

4.2. Next, suppose that $\Theta(z) = z^n$. An argument similar to the one above gives us yet another class of block Toeplitz operator matrices which are equivalent to truncated Toeplitz operators. This time, we take functions $\psi_i \in K_B$ ($-n \leq i \leq n-1$), and we assume that $A_{\psi_i}^B$ is bounded for $-(n-1) \leq i \leq n-1$, while $A_{\bar{B}\psi_i}^B$ is bounded for $-n \leq i \leq n-2$. Consider a symbol function of the form

$$h(z) = \sum_{m=-n}^{n-1} \psi_m(z) B^m \quad (4.2)$$

Then the operator $A_h^{B^n}$ is bounded and unitarily equivalent to the following block Toeplitz operator matrix, each entry of which is a truncated Toeplitz operator on K_B :

$$\begin{pmatrix} A_{\psi_0 + \bar{B}\psi_{-1}}^B & A_{\psi_{-1} + \bar{B}\psi_{-2}}^B & \cdots & A_{\psi_{-(n-1)} + \bar{B}\psi_{-n}}^B \\ A_{\psi_1 + \bar{B}\psi_0}^B & A_{\psi_0 + \bar{B}\psi_{-1}}^B & \cdots & A_{\psi_{-(n-2)} + \bar{B}\psi_{-(n-1)}}^B \\ \ddots & \ddots & \ddots & \ddots \\ A_{\psi_{n-1} + \bar{B}\psi_{n-2}}^B & A_{\psi_{n-2} + \bar{B}\psi_{n-3}}^B & \cdots & A_{\psi_0 + \bar{B}\psi_{-1}}^B \end{pmatrix} \quad (4.3)$$

If $A = (a_{i-j})_{i,j=1,\dots,n}$ is a classical Toeplitz matrix, then, by considering the function h in (4.2) with $\psi_i = a_i$ we obtain (noting that $A_{\bar{B}}^B = 0$) that $A_{\psi(B)}^{B^n}$ is unitarily equivalent to $A \otimes I_{K_B}$ for any inner function B . This is Theorem 5.8 of [4].

4.3. Suppose that B, Θ are arbitrary inner functions and that $\phi = \bar{z}\Theta$. As noted in Section 2, $A_\phi^\Theta = \tilde{k}_0^\Theta \odot k_0^\Theta$. On the other hand, $A_{z\phi}^\Theta = A_\Theta^\Theta = 0$. Therefore (4.1) implies that, if $\psi \in K_B$ and A_ψ^B is bounded, then $A_{\psi(\phi \circ B)}^{\Theta \circ B}$ is bounded and unitarily equivalent to $A_\psi^B \otimes (\tilde{k}_0^\Theta \odot k_0^\Theta)$.

The following lemma is well known (and easy to prove).

Lemma 4.2. *Two operators of rank one $x_i \odot y_i \in \mathcal{L}(\mathcal{E}_i)$, $i = 1, 2$, are unitarily equivalent if and only if the following are satisfied:*

- (1) $\dim \mathcal{E}_1 = \dim \mathcal{E}_2$;
- (2) $\|x_1\| \|y_1\| = \|x_2\| \|y_2\|$;
- (3) $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$.

Corollary 4.3. *Suppose that $R = x \odot y$ is a rank one operator on a Hilbert space \mathcal{E} with $\dim \mathcal{E} \geq 2$. The following are equivalent:*

- (i) *There exists an inner function Θ such that R is unitarily equivalent to a scalar multiple of $\tilde{k}_0^\Theta \odot k_0^\Theta$ acting in K_Θ .*
- (ii) *Either $R = 0$ or x, y are not colinear.*

Proof. Suppose that (i) is satisfied. Then $\dim K_\Theta = \dim \mathcal{E} \geq 2$. According to (2.2), we have

$$\frac{|\langle \tilde{k}_0^\Theta, k_0^\Theta \rangle|}{\|\tilde{k}_0^\Theta\| \cdot \|k_0^\Theta\|} = \frac{|\tilde{k}_0^\Theta(0)|}{(1 - |\Theta(0)|^2)^{-1}} = |\Theta'(0)|(1 - |\Theta(0)|^2).$$

The last expression can be 1 only if $\Theta(z) = z$, which contradicts $\dim K_\Theta \geq 2$. Thus it is < 1 , so k_0^Θ and \tilde{k}_0^Θ are not colinear. If $R \neq 0$, it follows from Lemma 4.2 that x and y are not colinear.

Conversely, take a nonzero operator of rank 1 $x \odot y$ acting on \mathcal{E} ; we may suppose that it has norm 1, and therefore that $\|x\| = \|y\| = 1$. Then if Θ is such that $\Theta(0) = 0$, $\Theta'(0) = \langle x, y \rangle$, and $\dim K_\Theta = \dim \mathcal{E}$, a second application of Lemma 4.2 implies that $x \odot y$ is unitarily equivalent to $\tilde{k}_0^\Theta \odot k_0^\Theta$. \square

Let us also note that if ψ is analytic, then $A_\psi^B = A_{P_\Theta \psi}^B$. Then Corollary 4.3, together with the comments preceding Lemma 4.2, lead to the following result.

Proposition 4.4. *If ψ is an analytic function, A_ψ^B is bounded, and R is a non-selfadjoint operator of rank one, then $A_\psi^B \otimes R$ is unitarily equivalent to a TTO.*

Question: What happens if R is selfadjoint? Then $A_\psi^B \otimes R$ is unitarily equivalent to a scalar multiple of $A_\psi^B \oplus 0$ (where the zero operator operates on a space whose dimension is equal to the dimension of the range of $I - R$). More about this in the next section.

5. DIRECT SUM WITH ZERO

Question 5.11 of [4] asks for conditions under which a direct sum of TTOs is unitarily equivalent to another TTO. We are interested here in the situation where one of the operators is 0 on a space of a given dimension k . The question then becomes: find classes of TTOs A and integers $k = 1, 2, \dots, \infty$ such that, if 0_k denotes the zero operator on a space of dimension k , then $A \oplus 0_k$ is unitarily equivalent to a TTO.

Two such classes of truncated Toeplitz operators have already been discovered:

- (1) A a rank one operator, k arbitrary (by [4, Theorem 5.1].)
- (2) A a normal operator, k arbitrary (by [4, Theorem 5.7].)

Below we present two more classes of pairs (A, k) that belong to the category described above. The first is obtained by using the techniques of the previous section.

Theorem 5.1. *Let B be an inner function, $\psi \in K_B$, $\zeta \in \mathbb{T}$, and $n \in \mathbb{N} \cup \{\infty\}$. If $d = \dim K_B$, then the operator $A_{\psi+\zeta\bar{B}\psi} \oplus 0_{nd}$ is unitarily equivalent to a TTO.*

Proof. Let Θ be an inner function whose angular derivative at ζ exists and such that $\dim K_\Theta = n + 1$. As noted in Section 2, $k_\zeta^\Theta \odot k_\zeta^\Theta$ is a TTO on K_Θ with symbol $k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1$.

Let $\phi = k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1 + \Theta(\zeta)\bar{\Theta}$; then ϕ is also a symbol of $k_\zeta^\Theta \odot k_\zeta^\Theta$ by (2.1), and a simple calculation shows that $\phi - \bar{\zeta}z\phi = -\overline{\Theta(\zeta)}\Theta + \Theta(\zeta)\bar{\Theta}$. Applying again (2.1), it follows that $A_\phi^\Theta = \bar{\zeta}A_{z\phi}^\Theta = k_\zeta^\Theta \odot k_\zeta^\Theta$. So, by using (3.1), we obtain that

$$A_{\psi(\phi \circ B)}^{\Theta \circ B} \omega_B = \omega_B \left(A_{\psi+\zeta\bar{B}\psi}^B \otimes (k_\zeta^\Theta \odot k_\zeta^\Theta) \right).$$

Since $k_\zeta^\Theta \odot k_\zeta^\Theta$ is a self adjoint operator of rank one, $\frac{1}{\|k_\zeta^\Theta\|^2} A_{\psi+\zeta\bar{B}\psi}^B \otimes (k_\zeta^\Theta \odot k_\zeta^\Theta)$ is unitarily equivalent to $A_{\psi+\zeta\bar{B}\psi} \oplus 0_d \oplus \dots \oplus 0_d$, where 0_d is repeated n times. Therefore the last operator is unitarily equivalent to the TTO defined on $K_{\Theta \circ B}$ and with symbol $\frac{1}{\|k_\zeta^\Theta\|^2} \psi \left(k_\zeta^\Theta \circ B + \overline{k_\zeta^\Theta \circ B} + \Theta(\zeta) \overline{\Theta \circ B} - 1 \right)$. \square

A particular case of Theorem 5.1 is worth mentioning: taking $\Theta(z) = z^2$, one obtains that the operator $A_{\psi+\zeta\bar{B}\psi} \oplus 0_d$ is unitarily equivalent to $A_{\psi(\phi \circ B)}^{B^2}$, where $\phi(z) = \frac{1}{2}(\bar{\zeta}^2 z^2 + \bar{\zeta} z + 1 + \zeta z)$.

The second result in this direction is obtained without using the previous sections; we have added it because it fits naturally in this context. We start with a simple lemma.

Lemma 5.2. *Suppose Θ is inner and $\phi \in H^\infty$. Then $K_\Theta \ominus \ker A_\phi^\Theta = K_u$ for some inner function u which satisfies $u|\Theta$ and $\Theta|u\phi$.*

Proof. We claim first that $\ker A_\phi^\Theta \oplus \Theta H^2$ is a z -invariant subspace of H^2 . Indeed, since ΘH^2 is z -invariant, what is left is to see that $z(\ker A_\phi^\Theta) \subseteq \ker A_\phi^\Theta \oplus \Theta H^2$. But $f \in \ker A_\phi^\Theta$ implies that $\phi f \in \Theta H^2$, whence $z\phi f \in \Theta H^2$. Thus, using the fact that $\phi(\Theta H^2) \subseteq \Theta H^2$, we have that:

$$A_\phi^\Theta(P_\Theta(zf)) = P_\Theta \phi P_\Theta(zf) = P_\Theta(\phi z f - \phi P_{\Theta H^2}(zf)) = 0,$$

which means that $P_\Theta(zf) \in \ker A_\phi^\Theta$ and thus $zf \in \ker A_\phi^\Theta \oplus \Theta H^2$.

It follows then that $\ker A_\phi^\Theta \oplus \Theta H^2 = uH^2$ for some inner function u , and thus

$$K_\Theta \ominus \ker A_\phi^\Theta = H^2 \ominus (\ker A_\phi^\Theta + \Theta H^2) = K_u.$$

It is immediate that $K_u \subseteq K_\Theta$ is equivalent to $u|\Theta$. As for the second divisibility property, note first that, since ϕ is analytic, $\phi(\Theta H^2) \subset \Theta H^2$. Also, if $f \in \ker A_\phi^\Theta$, then

$$\phi f = P_\Theta(\phi f) + P_{\Theta H^2}(\phi f) = A_\phi^\Theta f + P_{\Theta H^2}(\phi f) = P_{\Theta H^2}(\phi f) \in \Theta H^2.$$

Since $uH^2 = \ker A_\phi^\Theta \oplus \Theta H^2$, it follows that $\phi uH^2 \subset \Theta H^2$, whence $\Theta|u\phi$. \square

Theorem 5.3. *Suppose Θ is inner, $\phi \in H^\infty$, and $(A_\phi^\Theta)^2 = 0$. If $k = \dim K_\Theta \ominus \ker A_\phi^\Theta$, then $A_\phi^\Theta \oplus 0_k$ is unitarily equivalent to a TTO.*

Proof. Applying Lemma 5.2, we have $K_\Theta = K_u \oplus \ker A_\phi^\Theta$ for an inner function u with $u|\Theta$ and $\Theta|u\phi$. The matrix of A_ϕ^Θ corresponding to the orthogonal decomposition $K_\Theta = K_u \oplus \ker A_\phi^\Theta$ is

$$A_\phi^\Theta = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}. \quad (5.1)$$

Let us consider the TTO $A_{u\phi}^\Theta$ acting on $K_{u\Theta}$. We want to find its matrix according to the orthogonal decomposition

$$K_{u\Theta} = K_u \oplus uK_\Theta = K_u \oplus uK_u \oplus u(\ker A_\phi^\Theta).$$

First, if $f \in K_u$, then

$$A_{u\phi}^{u\Theta} f = P_{u\Theta}(u\phi f) = uP_{\Theta}(\bar{u}u\phi f) = uA_{\phi}^{\Theta} f.$$

Equation (5.1) shows us then that the first column of the matrix of $A_{u\phi}^{u\Theta}$ is $(0 \ 0 \ uA)^t$.

On the other hand, $\Theta|u\phi$ implies that $u\Theta|u^2\phi$, and therefore that $u\phi(uH^2) \subset u\Theta H^2$, whence the two remaining columns of the decomposition are formed only by zeros. Therefore the matrix of $A_{u\phi}^{u\Theta}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ uA & 0 & 0 \end{pmatrix}.$$

Finally, an operator unitarily equivalent to $A_{u\phi}^{u\Theta}$ is obtained through a permutation of the two first spaces (which have the same dimension, that of K_u). The matrix is then transformed into

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & uA & 0 \end{pmatrix},$$

which, again by (5.1), defines an operator unitarily equivalent to $A_{\phi}^{\Theta} \oplus 0_{\dim K_u}$. \square

Let us note that there is no loss of generality in supposing $\phi \in H^{\infty}$, rather than $\phi \in H^2$ and A_{ϕ}^{Θ} bounded. Indeed, the last assumptions imply, by the commutant lifting theorem, that A_{ϕ}^{Θ} has a symbol in H^{∞} . Also, a similar result obviously holds in the case where ϕ is antianalytic.

ACKNOWLEDGEMENTS

M. Zarrabi was partially supported by the ANR-09-BLAN-0058-01. D. Timotin thanks the University of Bordeaux for its hospitality during his stay in Bordeaux, which led to the completion of this paper.

REFERENCES

- [1] ALEKSANDROV, A.B. Invariant subspaces of shift operators. An axiomatic approach. *J. Soviet Math.* 22 (1983), 1695–1708.
- [2] BARANOV, A., BESSONOV, R., KAPUSTIN, V. Symbols of truncated Toeplitz operators. Preprint.
- [3] BARANOV, A., CHALENDAR, I., FRICAIN, E., MASHREGHI, J., TIMOTIN, D. Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators. *J. Functional Analysis*, to appear.
- [4] CIMA, J., GARCIA, S. ROSS, W. AND WOGEN, W. Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. *Indiana Univ. Math. J.*, to appear.
- [5] CIMA, J., MATHESON, A., ROSS, W., *The Cauchy Transform*, AMS, Providence, 2006.
- [6] CIMA, J., ROSS, W. AND WOGEN, W. Truncated Toeplitz operators on finite dimensional spaces. *Oper. Matrices* 2 (2008), 357–369.
- [7] COWEN, C. On equivalence of Toeplitz operators. *J. Operator Theory* 7 (1982), no. 1, 167–172.

- [8] NIKOLSKI, N. K. *Treatise on the shift operator*. Springer-Verlag, Berlin, 1986. Grundlehren der Mathematischen Wissenschaften, 273.
- [9] SARASON, D. Generalized interpolation in H^∞ . *Trans. Amer. Math. Soc.* 127 (1967), 179–203.
- [10] SARASON, D. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices* 1 (2007), 491–526.
- [11] SZ.-NAGY, B., FOIAS, C. *Analyse harmonique des opérateurs de l'espace de Hilbert*. Masson, 1967.

UNIVERSITÉ DE BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX UMR 5251, 351,
COURS DE LA LIBÉRATION, F-33405 TALENCE CEDEX, FRANCE
E-mail address: `Elizabeth.Strouse@math.u-bordeaux1.fr`

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST
014700, ROMANIA
E-mail address: `Dan.Timotin@imar.ro`

UNIVERSITÉ DE BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX UMR 5251, 351,
COURS DE LA LIBÉRATION, F-33405 TALENCE CEDEX, FRANCE
E-mail address: `Mohamed.Zarrabi@math.u-bordeaux1.fr`